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Research paper

# System stability analysis via a perturbation technique☆

Laigang Guo, Yufu Chen*

*School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, PR China*

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ABSTRACT

In this paper, a discriminant for judging the stability of differential dynamical systems is given. The discriminant is a rational expression in the coefficients of the original system. How to transform a differential dynamical system to the normal form is shown, which is the origin of the discriminant. This method is more simple and practical than the traditional Lyapunov method. Examples are presented to show the applicability of the methodology and the convenience of the discriminant.



## 1. Introduction

System stability is a very important problem in the field of natural science and engineering technology. Any actual system is always moving or working under a variety of occasional and ongoing disturbances. When the system is under disturbance, whether the system can safely keep the motion track or working condition, that is, the stability of the system is the most important consideration. The stability of a system includes the stability of the equilibrium state and the stability of any motion. The stability of a given motion can be transformed into the stability of the equilibrium point. When we analyze the stability of the equilibrium point, the stability of equilibrium point are defined as Lyapunov stability, uniform stability, asymptotic stability, uniform asymptotic stability, exponential asymptotic stability and global asymptotic stability. The best general references here are [1,2].

The study of nonlinear control systems has made some significant progress. For the further research, we review some common methods for the stability analysis of nonlinear control systems first. The first one, linearization method, is used to approximate the nonlinear system by using the linearized model. The linear methods include tangent approximation method and least square method. The linearization approximation is only for weakly nonlinear systems. The second one, phase plane method, is based on time domain analysis, and it is used to solve the first or the second order nonlinear ordinary differential equations. By graphic method, the motion of the first order and second order systems are transformed into the phase trajectory of the position. The phase trajectory drawing method is simple and the calculation is small, and it is especially suitable for the analysis of the nonlinear system. The third one, the description of the function method, makes the nonlinear element approximate to a linear element, so that the stability of the system can be distinguished by using the Nyquist stability criterion. This method is mainly used to study the stability and self oscillation of nonlinear systems. For example, if the system produces self oscillation, then the method can be used to find out the frequency and amplitude

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* Corresponding author.
*E-mail address:* yfchen@ucas.ac.cn (Y. Chen).

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of the oscillation, and to find ways to eliminate the self oscillation. But it can not directly give reliable information about the transient response. The fourth one, the Lyapunov first method, is a method for studying the stability of an approximate mathematical model (linear model) of dynamic system. Its basic ideas include that: finding the balance state of the system; linearizing the state equation near the equilibrium points (including different equilibrium points); finding the characteristic values of the state equations with linearization; determining the stability of the system in the case of zero input. If the characteristic value is 0, we need to use the center manifold theorem. The idea of the stability criterion of Lyapunov first method is the same as the classical control theory. We need to solve the eigenvalues of the linearized state equations or the linear state equations, and analyze the stability based on the eigenvalues in the complex plane. Because the Lyapunov first method requires the eigenvalues of the system after linearization, so the method can only be applied to the nonlinear constant system, linear constant system or weakly nonlinear problem, and can not be extended to time varying systems. The fifth one, the Lyapunov second method, is proposed by considering the limitations of the Lyapunov first method. The Lyapunov second method, also known as direct method, is applied for linear systems and nonlinear systems, constant systems and time varying systems. It is established on the basis of the stability analysis from the viewpoint of energy. Based on this view, if only we can find a positive energy function as a reasonable description of dynamic system of n-dimensional state, then we can judge the stability of the equilibrium state of the system by investigating whether the function decays over time. According to the characteristics of nonlinear system dynamic equations, we can find $V(x)$ by the correlation method, and determine the stability of nonlinear systems by the properties of $V(x)$ and $\dot{V}(x)$. The best general references here are [1,3–5].

Recently, many researchers have paid attention to the development of the Lyapunov method [6–10]. In addition, the method for studying the stability of a nonlinear system with the center manifold and normal forms theory appeared in 1985. The classical work can be seen in [11–15].

In this paper, we will consider the system which is characterized by the Jacobian matrix having a pair of purely imaginary and other hyperbolic eigenvalues at the equilibrium 0. Hyperbolic eigenvalues are the eigenvalues which have non-zero real parts. We propose a discriminant to judge the stability of a 3-dimensional analytic system. The discriminant is derived from a kind of normal forms of the system. We also show how to transform a 3-dimensional analytic system to the normal form. Then our results are generalized partially to n-dimensional system. Finally, examples are presented to show the applicability of the technique.

## 2. Stability analysis of 3-dimensional system

In this section, we will give a theorem for the stability analysis of 3-dimensional system. Our main work is to prove the theorem. The process described here can be implemented on a computer with Maple or Mathematica. First, we consider such a system

$$\dot{x} = f(x), \quad x \in R^n, \tag{1}$$

where $f$ is analytic, and $x = 0$ is an equilibrium of system (1), i.e. $f(0) = 0$. System (1) is linearized to

$$\dot{x} = Jx + F(x), \quad x \in R^n. \tag{2}$$

This function $F$ and its first order derivative vanish at $x = 0$. We can assume that $J$ is Jordan canonical form (otherwise we change $J$ to Jordan canonical form by a linear transformation $x = Ty$). Here, $J$ has a pair of purely imaginary eigenvalues $\pm i\omega_c$ at the equilibrium $x = 0$. Without loss of generality, we assume $\omega_c = 1$ (otherwise one may use an additional transformation $t' = w_c t$) to change frequency $\omega_c$ to 1), and the Jordan canonical form of the Jacobian matrix of system (2) at $x = 0$ is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}, \quad A \in R^{(n-2)\times(n-2)},$$

where $A$ is hyperbolic. For most of the physical situations, we can assume the unstable manifold is empty (the eigenvalues of $A$ only have negative real part).

In the 3-dimensional case, to begin with, we write Eq. (2) in the component form,

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_1, x_2, x_3), \\ \dot{x}_2 &= -x_1 + f_2(x_1, x_2, x_3), \\ \dot{x}_3 &= -\alpha_3 x_3 + f_3(x_1, x_2, x_3), \end{aligned} \tag{3}$$

where $\alpha_3 > 0$, and $f_i(0, 0, 0) = 0, \ \partial f_i(0, 0, 0)/\partial x_j = 0, \ i, j = 1, 2, 3$. In order to give the theorem easily, we write further (3) as follows,

$$\begin{aligned} \dot{x}_1 &= x_2 + p_1 {x_1}^2 + p_2 {x_2}^2 + p_3 x_1 x_2 + p_4 x_1 x_3 + p_5 x_2 x_3 + p_6 {x_1}^3 + g_1(x_1, x_2, x_3), \\ \dot{x}_2 &= -x_1 + q_1 {x_1}^2 + q_2 {x_2}^2 + q_3 x_1 x_2 + q_4 x_1 x_3 + q_5 x_2 x_3 + q_6 {x_2}^3 + g_2(x_1, x_2, x_3), \\ \dot{x}_3 &= -\alpha_3 x_3 + r_1 {x_1}^2 + r_2 {x_2}^2 + r_3 x_1 x_2 + g_3(x_1, x_2, x_3), \end{aligned} \tag{4}$$

where $g_1$ does not contain the terms like ${x_1}^2$, ${x_2}^2$, $x_1x_2$, $x_1x_3$, $x_2x_3$, ${x_1}^3$, $g_2$ does not contain the terms like ${x_1}^2$, ${x_2}^2$, $x_1x_2$, $x_1x_3$, $x_2x_3$, ${x_2}^3$, and $g_3$ does not contain the terms like ${x_1}^2$, ${x_2}^2$, $x_1x_2$.

**Theorem 1.** *For the system (4), if* $\Delta < 0$*, then the system will be unstable at the origin; if* $\Delta > 0$*, then the system will be asymptotically stable at the origin, where*

$$\Delta = (p_3 - 2q_1)p_1 + (p_3 + 2q_2)p_2 - (q_1 + q_2)q_3 - 3(p_6 + q_6) + \frac{(p_5 + q_4)(2r_1 - 2r_2 - \alpha_3 r_3)}{\alpha_3{}^2 + 4} + \frac{(-p_4 + q_5)(\alpha_3 r_1 - \alpha_3 r_2 + 2r_3)}{\alpha_3{}^2 + 4} - \frac{2(p_4 + q_5)(r_1 + r_2)}{\alpha_3}.$$

**Proof.** We need to use the perturbation analysis method based on the method of multiple scales [16,17], which is frequently used for analyzing second-order nonlinear differential equations [18], usually given in the form $\ddot{x} + x = \varepsilon f(x, \dot{x})$, where dot indicates the differentiation with respect to time $t$, and $\varepsilon$ is a small parameter ($0 < \varepsilon \ll 1$), $f$ is a nonlinear analytic function and thus can be expressed in a Taylor series.Then we begin with introducing new independent variables according to

$$T_k = \varepsilon^k t, \;\; k = 0, 1, 2, \ldots$$

and

$$\frac{d}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \frac{dT_2}{dt}\frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \tag{5}$$

where the differential operator $D_k = \partial/\partial T_k$.

Then, we suppose that the solution of (3) in the neighborhood of $x = 0$ is represented by an expansion of the form

$$x_i(t; \varepsilon) = \varepsilon x_{i1}(T_0, T_1, \ldots) + \varepsilon^2 x_{i2}(T_0, T_1, \ldots) + \cdots, \;\; i = 1, 2, 3. \tag{6}$$

Note that the perturbation parameter $\varepsilon$ used in (6) is the same as that used in the time scales $T_k = \varepsilon^k t, \; k = 0, 1, 2, \ldots$.

Next, we substitute (5), (6) into (3), and balance the like powers of $\varepsilon$ results in the ordered perturbation equations

$$\varepsilon^1 : D_0 x_{11} = x_{21}, D_0 x_{21} = -x_{11}, D_0 x_{31} = -\alpha_3 x_{31}, \tag{7}$$

$$\begin{aligned} &\varepsilon^2 : D_0 x_{12} = x_{22} - D_1 x_{11} + f_{12}(x_{11}, x_{21}, x_{31}), \\ &D_0 x_{22} = -x_{12} - D_1 x_{21} + f_{22}(x_{11}, x_{21}, x_{31}), \\ &D_0 x_{32} = -\alpha_3 x_{32} + f_{32}(x_{11}, x_{21}, x_{31}), \end{aligned} \tag{8}$$

where $f_{i2} = (d^2/d\varepsilon^2)[f_i(x_1, x_2, x_3)]_{\varepsilon=0}$. $f_{i2}$ is the function of $x_{i1} (i = 1, 2, 3)$ which has been solved from (7). In general, function $f_{ik}$ only involves variables which have been solved from the previous (k-1) steps perturbation equations. To solve the $\varepsilon^1$ order Eq. (7), these equations will be divided into two groups, one of which consists of the first two equations, and the other one includes the remaining equations.

From the first group, we find that

$$\begin{aligned} &D_0^2 x_{11} + x_{11} = 0, \;\; x_{11} = r(T_1, T_2, \ldots)\cos[T_0 + \varphi(T_1, T_2, \ldots)] = r\cos\theta, \\ &x_{21} = D_0 x_{11} = -r(T_1, T_2, \ldots)\sin[T_0 + \varphi(T_1, T_2, \ldots)] = -r\sin\theta. \end{aligned}$$

Because we only care about the asymptotic behavior of the system, the solutions of the second group are contributed from the first two variables $x_1$ and $x_2$ only. The asymptotic $\varepsilon^1$ solutions of the second group are given by $x_{31} = 0$. Substituting $x_{11}$, $x_{21}$, $x_{31}$ into (8), thus we can solve Eq. (8). From the first two equations of (8), we get an equation

$$D_0{}^2 x_{12} + x_{12} = -D_1 D_0 x_{11} - D_1 x_{21} + D_0 f_{12} + f_{22}. \tag{9}$$

Substituting $x_{11}$, $x_{21}$, $x_{31}$ into the right side of Eq. (9) gives an expression in terms of trigonometric functions $\cos k\theta$ and $\sin k\theta, \;\; k = 0, 1, 2$. To eliminate possible secular terms which may appear in $x_{12}$, it is required that the coefficients of the two terms $\cos k\theta$ and $\sin k\theta$ equal zero, which in turn yields the explicit expression for $D_1 r, D_1\varphi$ and $x_{12}, x_{22}$. Follow this way, we also get $x_{1i}, \; x_{2i}, \; x_{3i}, \; D_{i-1}r, \; D_{i-1}\varphi \; (i > 1)$ from the $\varepsilon^i$ order perturbation equations.

This method can be used to solve any higher order perturbation equations of (3), finally, it is the result that

$$\begin{aligned} &x_1 = r\cos\theta + h_1(r\cos\theta, -r\sin\theta), \\ &x_2 = -r\sin\theta + h_2(r\cos\theta, -r\sin\theta), \\ &x_3 = h_3(r\cos\theta, -r\sin\theta), \end{aligned} \tag{10}$$

and we also find that

$$\begin{aligned} &\dot{r} = \frac{\partial r}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial r}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial r}{\partial T_2}\frac{\partial T_2}{\partial t} + \cdots = D_0 r + \varepsilon D_1 r + \varepsilon^2 D_2 r + \cdots, \\ &\dot{\theta} = 1 + \frac{\partial \varphi}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial \varphi}{\partial T_1}\frac{\partial T_1}{\partial t} + \cdots = 1 + D_0\varphi + \varepsilon D_1\varphi + \cdots. \end{aligned}$$

We use a back scaling $\varepsilon r \to r$, and find that

$$\begin{aligned} &\dot{r} = a_{13}r^3 + a_{15}r^5 + \cdots + a_{1(2n+1)}r^{2n+1} + \cdots, \\ &\dot{\theta} = 1 + a_{23}r^2 + a_{25}r^4 + \cdots + a_{2(2n+1)}r^{2n} + \cdots. \end{aligned} \tag{11}$$

One has got the simplest normal form of (11) in [11], in this article we just discuss one case that (11) can be reduced to the following form,

$$\begin{cases}\dot{\rho} = a_{13}\rho^3 + a_{15}\rho^5, \\ \dot{\varphi} = 1 + a_{23}\rho^2, \end{cases} \quad if\ a_{13} \neq 0. \tag{12}$$

Note that $a_{13}$ in (11) is the same as that in (12). The next work is to prove that (3) can be reduced to (11) by the center manifold theory. If the transformation

$$y_1 = r\cos\theta,\ \ y_2 = -r\sin\theta$$

is introduced into (10), one can rewrite Eq. (10) as follows,

$$\begin{aligned} x_1 &= y_1 + h_1(y_1, y_2), \\ x_2 &= y_2 + h_2(y_1, y_2), \\ x_3 &= h_3(y_1, y_2). \end{aligned} \tag{13}$$

Let us give the following near-identity transformation to (3):

$$\begin{aligned} x_1 &= y_1 + h_1(y_1, y_2), \\ x_2 &= y_2 + h_2(y_1, y_2), \\ x_3 &= y_3, \end{aligned} \tag{14}$$

then we have

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{bmatrix} 1 + \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & 0 \\ \frac{\partial h_2}{\partial y_1} & 1 + \frac{\partial h_2}{\partial y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix},$$

and

$$\begin{aligned} \begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix} &= \begin{bmatrix} 1 + \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & 0 \\ \frac{\partial h_2}{\partial y_1} & 1 + \frac{\partial h_2}{\partial y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} \\ &= \begin{bmatrix} 1 + \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & 0 \\ \frac{\partial h_2}{\partial y_1} & 1 + \frac{\partial h_2}{\partial y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{pmatrix} x_2 + f_1(x_1, x_2, x_3) \\ -x_1 + f_2(x_1, x_2, x_3) \\ -\alpha_3 x_3 + f_3(x_1, x_2, x_3) \end{pmatrix}. \end{aligned} \tag{15}$$

We can obtain the system which has the same stability properties as (3) at the origin by substituting (14) into (15), as follows,

$$\begin{aligned} \dot{y}_1 &= y_2 + e_1(y_1, y_2, y_3), \\ \dot{y}_2 &= -y_1 + e_2(y_1, y_2, y_3), \\ \dot{y}_3 &= e_3(y_1, y_2, y_3). \end{aligned} \tag{16}$$

Comparing (13) and (14), we get $y_3 = h_3(y_1, y_2)$. And $y_3 = h_3(y_1, y_2)$ is the center manifold of system (16), because of the related theory of center manifold in [19]. Substituting $y_3 = h_3(y_1, y_2)$ into the first two equations of (16), we can get the reduced system as follows,

$$\begin{aligned} \dot{y}_1 &= y_2 + a_{13}({y_1}^2 + {y_2}^2)y_1 + \cdots + a_{1(2n+1)}({y_1}^2 + {y_2}^2)^n y_1 + \cdots + a_{23}({y_1}^2 + {y_2}^2)y_2 + \cdots \\ &\quad + a_{2(2n+1)}({y_1}^2 + {y_2}^2)^n y_2 + \cdots \\ \dot{y}_2 &= -y_1 + a_{13}({y_1}^2 + {y_2}^2)y_2 + \cdots + a_{1(2n+1)}({y_1}^2 + {y_2}^2)^n y_2 + \cdots - a_{23}({y_1}^2 + {y_2}^2)y_1 - \cdots \\ &\quad - a_{2(2n+1)}({y_1}^2 + {y_2}^2)^n y_1 - \cdots. \end{aligned} \tag{17}$$

And the reduced system is obtained in [20,21] too. The system (17) given in polar coordinates can be written as (11). The best general references here are [20,21]. Thus, according to the center manifold theory, (11) and (3) have the same stability properties at the origin. As we know, in (11), if $a_{13} > 0$, the system is unstable, if $a_{13} < 0$, the system is asymptotically stable.

Thus the next work is to solve $a_{13}$. According to the calculations above, we get $a_{13}r^3 = D_2r$. Then the next work is to solve $D_2r$, and we notice that $D_2r$ appears in $\varepsilon^3$ order perturbation equations,

$$\begin{aligned}&\varepsilon^3 : D_0x_{13} + D_1x_{12} + D_2x_{11} = x_{23} + f_{13},\\ &D_0x_{23} + D_1x_{22} + D_2x_{21} = -x_{13} + f_{23},\\ &D_0x_{33} + D_1x_{32} = -\alpha_3x_{33} + f_{33}.\end{aligned} \tag{18}$$

Note that, $f_{i3} = (d^3/d\varepsilon^3)[f_i(x_1, x_2, x_3)]_{\varepsilon=0}$, $i = 1, 2, 3$. Substituting (6) into (3), we can know that $f_{i3}$ is in terms of $x_{11}{}^3$, $x_{21}{}^3$, $x_{11}x_{i2}$, $x_{21}x_{j2}$, $(i, j = 1, 2, 3)$. Next,we will solve $x_{12}$,$x_{22}$,$x_{32}$. Comparing (3) and (4),we find that

$$\begin{aligned}f_1(x_1, x_2, x_3) &= p_1x_1{}^2 + p_2x_2{}^2 + p_3x_1x_2 + p_4x_1x_3 + p_5x_2x_3 + p_6x_1{}^3 + g_1(x_1, x_2, x_3),\\ f_2(x_1, x_2, x_3) &= q_1x_1{}^2 + q_2x_2{}^2 + q_3x_1x_2 + q_4x_1x_3 + q_5x_2x_3 + q_6x_2{}^3 + g_2(x_1, x_2, x_3),\\ f_3(x_1, x_2, x_3) &= r_1x_1{}^2 + r_2x_2{}^2 + r_3x_1x_2 + g_3(x_1, x_2, x_3).\end{aligned} \tag{19}$$

In (9), $D_0{}^2x_{12} + x_{12} = -D_1D_0x_{11} - D_1x_{21} + D_0f_{12} + f_{22}$, by calculations, we find $D_1r = D_1\varphi = 0$, which yields that

$$D_0{}^2x_{12} + x_{12} = D_0f_{12} + f_{22}. \tag{20}$$

According to previous definitions, $f_{12}$, $f_{22}$, $f_{32}$ are in terms of $x_{11}{}^2$, $x_{21}{}^2$, $x_{11}x_{21}$, then we learn from (6) and (19) that

$$\begin{aligned}f_{12}(x_{11}, x_{21}, x_{31}) &= p_1x_{11}{}^2 + p_2x_{21}{}^2 + p_3x_{11}x_{21},\\ f_{22}(x_{11}, x_{21}, x_{31}) &= q_1x_{11}{}^2 + q_2x_{21}{}^2 + q_3x_{11}x_{21},\\ f_{32}(x_{11}, x_{21}, x_{31}) &= r_1x_{11}{}^2 + r_2x_{21}{}^2 + r_3x_{11}x_{21}.\end{aligned} \tag{21}$$

Substituting the first two equations of (21) and $x_{11}$, $x_{21}$ into (20), we find that

$$\begin{aligned}D_0^2x_{12} + x_{12} &= D_0f_{12} + f_{22} = D_0(p_1x_{11}{}^2 + p_2x_{21}{}^2 + p_3x_{11}x_{21}) + q_1x_{11}{}^2 + q_2x_{21}{}^2 + q_3x_{11}x_{21},\\ D_0{}^2x_{12} + x_{12} &= \left(p_2 - p_1 - \frac{q_3}{2}\right)r^2\sin 2\theta + \left(\frac{q_1}{2} - \frac{q_2}{2} - p_3\right)r^2\cos 2\theta + \left(\frac{q_1}{2} + \frac{q_2}{2}\right)r^2.\end{aligned} \tag{22}$$

Because the maximum degree in the right of (22) is two , we can assume that

$$x_{12} = A_0 + A_1\cos\theta + B_1\sin\theta + A_2\cos 2\theta + B_2\sin 2\theta. \tag{23}$$

Substituting (23) into (22), and using the harmonic balance method [22], the corresponding terms of both sides are balanced, we find that

$$\begin{aligned}x_{12} &= A_{10}r^2 + A_{11}r^2\cos 2\theta + A_{12}r^2\sin 2\theta\\ &= \frac{q_1 + q_2}{2}r^2 + \frac{q_2 + 2p_3 - q_1}{6}r^2\cos\theta + \frac{q_3 + 2p_1 - 2p_2}{6}r^2\sin 2\theta.\end{aligned}$$

From the first equation of (8), we find

$$\begin{aligned}x_{22} &= D_0x_{12} + D_1x_{11} - f_{12} = A_{20}r^2 + A_{21}r^2\cos 2\theta + A_{22}r^2\sin 2\theta\\ &= -\frac{p_1 + p_2}{2}r^2 + \frac{p_1 - p_2 + 2q_3}{6}r^2\cos 2\theta + \frac{2q_1 - 2q_2 - p_3}{6}r^2\sin 2\theta.\end{aligned}$$

From the third equation in (8), we get

$$D_0x_{32} + \alpha_3x_{32} = f_{32}(x_{11}, x_{21}, x_{31}). \tag{24}$$

Substituting $x_{32} = A_0 + A_1\cos\theta + B_1\sin\theta + A_2\cos 2\theta + B_2\sin 2\theta$ and $f_{32}$ into (24), we can find

$$\begin{aligned}x_{32} &= A_{30}r^2 + A_{31}r^2\cos 2\theta + A_{32}r^2\sin 2\theta\\ &= \frac{r_1 + r_2}{2\alpha_3}r^2 + \frac{\alpha_3r_1 - \alpha_3r_2 + 2r_3}{2(\alpha_3{}^2 + 4)}r^2\cos 2\theta + \frac{2r_1 - 2r_2 - \alpha_3r_3}{2(\alpha_3{}^2 + 4)}r^2\sin 2\theta.\end{aligned}$$

Because $D_1r = D_1\varphi = 0$, we can get the following equations from the first two equations of (18),

$$\begin{aligned}D_0^2x_{13} + x_{13} &= -D_0D_1x_{12} - D_0D_2x_{11} - D_1x_{22} - D_2x_{21} + f_{23} + D_0f_{13},\\ D_0{}^2x_{13} + x_{13} &= 2(D_2r\cdot\sin\theta + rD_2\varphi\cdot\cos\theta) + f_{23} + D_0f_{13}.\end{aligned} \tag{25}$$

To eliminate the possible secular terms which may appear in $x_{13}$, it is required that the coefficients of the two terms $\cos\theta$ and $\sin\theta$ equal 0 in (25), and we can see that $D_2r$ only appears as the coefficient of $\sin\theta$, thus we only need to search the terms about $\sin\theta$ in $f_{23}$ and $D_0f_{13}$, which are the terms about $\sin\theta$ in $f_{23}$ and $\cos\theta$ in $f_{13}$. Because $f_{i3}$ is in terms of $x_{11}{}^3$, $x_{21}{}^3$, $x_{11}x_{i2}$, $x_{21}x_{j2}$, $(i, j = 1, 2, 3)$, in (19), we can assume

$$\begin{aligned}f_{13} &= 2p_1x_{11}x_{12} + p_3x_{11}x_{22} + p_4x_{11}x_{32} + p_3x_{21}x_{12} + 2p_2x_{21}x_{22} + p_5x_{21}x_{32} + p_6x_{11}{}^3 + f_{131},\\ f_{23} &= 2q_1x_{11}x_{12} + q_3x_{11}x_{22} + q_4x_{11}x_{32} + q_3x_{21}x_{12} + 2q_2x_{21}x_{22} + q_5x_{21}x_{32} + q_6x_{21}{}^3 + f_{231},\end{aligned} \tag{26}$$

where $f_{131} = c_1 x_{21}^3$, $f_{231} = c_2 x_{11}^3$, $c_1, c_2$ is constant. Substituting (26) and $x_{12}, x_{22}, x_{32}$ into (25), it results that

$$\begin{aligned} a_{13} r^3 &= D_2 r \\ &= -\frac{1}{2} r^3 \Big[ (-2p_1 - q_3) A_{10} + \left( \frac{1}{2} q_3 - p_1 \right) A_{11} - (p_3 + 2q_2) A_{20} + \left( -\frac{1}{2} p_3 + q_2 \right) A_{21} - (p_4 + q_5) A_{30} \\ &\quad + \frac{1}{2} (q_5 - p_4) A_{31} + \left( \frac{1}{2} p_3 + q_1 \right) A_{12} + \left( p_2 + \frac{1}{2} q_3 \right) A_{22} + \left( \frac{1}{2} p_5 + \frac{1}{2} q_4 \right) A_{32} - \frac{3}{4} p_6 - \frac{3}{4} q_6 \Big]. \end{aligned}$$

Substituting $A_{i0}, A_{i1}, A_{i2}$ $(i = 1, 2, 3)$ into the above equation, we find

$$\begin{aligned} a_{13} = -\frac{1}{8} \Big[ & (p_3 - 2q_1) p_1 + (p_3 + 2q_2) p_2 - (q_1 + q_2) q_3 - 3(p_6 + q_6) + \frac{(p_5 + q_4)(2r_1 - 2r_2 - \alpha_3 r_3)}{\alpha_3^2 + 4} \\ & + \frac{(-p_4 + q_5)(\alpha_3 r_1 - \alpha_3 r_2 + 2r_3)}{\alpha_3^2 + 4} - \frac{2(p_4 + q_5)(r_1 + r_2)}{\alpha_3} \Big]. \end{aligned}$$

Then, $\Delta = -8a_{13}$, thus the theorem is obtained. □

## 3. Stability analysis of n-dimensional system

In this section, we mainly give three theorems to solve the center manifold of system (2) which is n-dimensional system, and solve the normal forms of the reduced system which is obtained by the center manifold theory. Then we give a discriminant which is solved by the coefficients of the system to judge the stability of the n-dimensional system by the normal forms. According to the previous discussion, in the system (2),

$$\dot{x} = Jx + F(x), x \in R^n,$$

where F and its first derivative vanish at $x = 0$, and

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & A \end{bmatrix}, \ A \in R^{(n-2) \times (n-2)},$$

the eigenvalues of $A$ are hyperbolic, and their real parts are negative. Thus, we assume that

$$A = \begin{bmatrix} B & & & \\ & C & & \\ & & D & \\ & & & E \end{bmatrix},$$

$$B = \begin{bmatrix} -\alpha_1 & 1 & & & & & & \\ & -\alpha_1 & 1 & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & 1 & & \\ & & & & & -\alpha_1 & & \\ & & & & & & -\alpha_1 & & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \\ & & & & & & & & & -\alpha_1 \end{bmatrix}_{(k_2 \times k_2)},$$

$$C = \begin{bmatrix} -\alpha_2 & & & & \\ & -\alpha_3 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & -\alpha_{k_3} \end{bmatrix},$$

$$D=\begin{bmatrix} -\alpha & \omega & 1 & & & & & & & & & \\ -\omega & -\alpha & & 1 & & & & & & & & \\ & & -\alpha & \omega & 1 & & & & & & & \\ & & -\omega & -\alpha & & 1 & & & & & & \\ & & & & \cdot & & \cdot & & & & & \\ & & & & & \cdot & & 1 & & & & \\ & & & & & & -\alpha & \omega & & & & \\ & & & & & & -\omega & -\alpha & & & & \\ & & & & & & & & -\alpha & \omega & & \\ & & & & & & & & -\omega & -\alpha & & \\ & & & & & & & & & & \cdot & \\ & & & & & & & & & & & \cdot \\ & & & & & & & & & & & -\alpha \quad \omega \\ & & & & & & & & & & & -\omega \quad -\alpha \end{bmatrix}_{(k_5-k_3)\times(k_5-k_3)},$$

$$E=\begin{bmatrix} -\alpha_{k_5+1} & \omega_{k_5+1} & & & & & & \\ -\omega_{k_5+1} & -\alpha_{k_5+1} & & & & & & \\ & & -\alpha_{k_5+2} & \omega_{k_5+2} & & & & \\ & & -\omega_{k_5+2} & -\alpha_{k_5+2} & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & -\alpha_{n-1} & \omega_{n-1} \\ & & & & & & -\omega_{n-1} & -\alpha_{n-1} \end{bmatrix}.$$

Note that C has no multiple real eigenvalues, and E has no multiple complex eigenvalues. B has multiple real eigenvalues, the multiplicity of 1 in B is $k_1$. D has multiple complex eigenvalues, the multiplicity of 1 in D is $k_4 - k_3$ . Here, the multiplicity of 1 can be obtained with the knowledge of the Jordan canonical form. Then, we write the system (2) in the component form

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1, x_2, \ldots, x_n),\\
\dot{x}_2 &= -x_1 + f_2(x_1, x_2, \ldots, x_n),\\
\dot{x}_a &= -\alpha_1 x_a + x_{a+1} + f_a(x_1, x_2, \ldots, x_n), \ (a = 3, 4, \ldots, k_1 + 2),\\
\dot{x}_b &= -\alpha_1 x_b + f_b(x_1, x_2, \ldots, x_n), \ (b = k_1 + 3, \ldots, k_2 + 2),\\
\dot{x}_p &= -\alpha_p x_p + f_p(x_1, x_2, \ldots, x_n), \ (p = k_2 + 3, \ldots, k_3),\\
\dot{x}_c &= -\alpha x_c + \omega x_{c+1} + x_{c+2} + f_c(x_1, x_2, \ldots, x_n),\\
\dot{x}_{c+1} &= -\omega x_c - \alpha x_{c+1} + x_{c+3} + f_{c+1}(x_1, x_2, \ldots, x_n), \ (c = k_3 + 1, k_3 + 3, \ldots, k_4 - 1),\\
\dot{x}_d &= -\alpha x_d + \omega x_{d+1} + f_d(x_1, x_2, \ldots, x_n),\\
\dot{x}_{d+1} &= -\omega x_d - \alpha x_{d+1} + f_{d+1}(x_1, x_2, \ldots, x_n), \ (d = k_4 + 1, k_4 + 3, \ldots, k_5 - 1),\\
\dot{x}_q &= -\alpha_q x_q + \omega_q x_{q+1} + f_q(x_1, x_2, \ldots, x_n),\\
\dot{x}_{q+1} &= -\omega_q x_q - \alpha_q x_{q+1} + f_{q+1}(x_1, x_2, \ldots, x_n), \ (q = k_5 + 1, k_5 + 3, \ldots, n - 1),
\end{aligned} \tag{27}$$

where $\alpha_1 > 0, \alpha_p > 0, \alpha > 0, \alpha_q > 0, \omega > 0, \omega_q > 0$.

And we assume that $f_i = s_{i1}{x_1}^2 + s_{i2}{x_2}^2 + f_{i0},\ i = 1, 2, \ldots, n.$ $f_{i0}$ does not contain the terms like ${x_1}^2$ and ${x_2}^2$. We take the approach used in studying the 3-dimensional systems, namely the perturbation analysis and multiple scale method, to study the n-dimensional system. Here, we only write $\varepsilon^1$ and $\varepsilon^2$ order equations.

$$\begin{aligned}
&\varepsilon^1:\\
&D_0 x_{11} = x_{21},\\
&D_0 x_{21} = -x_{11},\\
&D_0 x_{a1} = -\alpha_1 x_{a1} + x_{(a+1)1}, \ (a = 3, 4, \ldots, k_1 + 2),\\
&D_0 x_{b1} = -\alpha_1 x_{b1}, \ (b = k_1 + 3, \ldots, k_2 + 2),\\
&D_0 x_{p1} = -\alpha_p x_{p1}, \ (p = k_2 + 3, \ldots, k_3),\\
&D_0 x_{c1} = -\alpha x_{c1} + \omega x_{(c+1)1} + x_{(c+2)1},
\end{aligned}$$

$$
\begin{aligned}
&D_0x_{(c+1)1} = -\omega x_{c1} - \alpha x_{(c+1)1} + x_{(c+3)1}, (c = k_3+1, k_3+3, \ldots, k_4-1),\\
&D_0x_{d1} = -\alpha x_{d1} + \omega x_{(d+1)1},\\
&D_0x_{(d+1)1} = -\omega x_{d1} - \alpha x_{(d+1)1}, (d = k_4+1, k_4+3, \ldots, k_5-1),\\
&D_0x_{q1} = -\alpha_q x_{q1} + \omega_q x_{(q+1)1},\\
&D_0x_{(q+1)1} = -\omega_q x_{q1} - \alpha_q x_{(q+1)1}, (q = k_5+1, k_5+3, \ldots, n-1),
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&\varepsilon^2:\\
&D_0x_{12} + D_1x_{11} = x_{22} + f_{12}(x_{11}, x_{21}, \ldots, x_{n1}),\\
&D_0x_{22} + D_1x_{21} = -x_{12} + f_{22}(x_{11}, x_{21}, \ldots, x_{n1}),\\
&D_0x_{a2} = -\alpha_1 x_{a2} + x_{(a+1)2} + f_{a2}(x_{11}, x_{21}, \ldots, x_{n1}), (a = 3, 4, \ldots, k_1+2),\\
&D_0x_{b2} = -\alpha_1 x_{b2} + f_{b2}(x_{11}, x_{21}, \ldots, x_{n1}), (b = k_1+3, \ldots, k_2+2),\\
&D_0x_{p2} = -\alpha_p x_{p2} + f_{p2}(x_{11}, x_{21}, \ldots, x_{n1}), (p = k_2+3, \ldots, k_3),\\
&D_0x_{c2} = -\alpha x_{c2} + \omega x_{(c+1)2} + x_{(c+2)2} + f_{c2}(x_{11}, x_{21}, \ldots, x_{n1}),\\
&D_0x_{(c+1)2} = -\omega x_{c2} - \alpha x_{(c+1)2} + x_{(c+3)2} + f_{(c+1)2}(x_{11}, x_{21}, \ldots, x_{n1}), (c = k_3+1, k_3+3, \ldots, k_4-1),\\
&D_0x_{d2} = -\alpha x_{d2} + \omega x_{(d+1)2} + f_{d2}(x_{11}, x_{21}, \ldots, x_{n1}),\\
&D_0x_{(d+1)2} = -\omega x_{d2} - \alpha x_{(d+1)2} + f_{(d+1)2}(x_{11}, x_{21}, \ldots, x_{n1}), (d = k_4+1, k_4+3, \ldots, k_5-1),\\
&D_0x_{q2} = -\alpha_q x_{q2} + \omega_q x_{(q+1)2} + f_{q2}(x_{11}, x_{21}, \ldots, x_{n1}),\\
&D_0x_{(q+1)2} = -\omega_q x_{q2} - \alpha_q x_{(q+1)2} + f_{(q+1)2}(x_{11}, x_{21}, \ldots, x_{n1}), (q = k_5+1, k_5+3, \ldots, n-1).
\end{aligned} \tag{29}
$$

The equations of (28) can be divided into two groups, one of which consists of the first two equations, and the other one includes the remaining equations. According to the first group, we obtain

$$
\begin{aligned}
&D_0^2x_{11} + x_{11} = 0,\ \ x_{11} = r(T_1, T_2, \ldots)\cos[T_0 + \varphi(T_1, T_2, \ldots)] = r\cos\theta,\\
&x_{21} = D_0x_{11} = -r(T_1, T_2, \ldots)\sin[T_0 + \varphi(T_1, T_2, \ldots)] = -r\sin\theta.
\end{aligned}
$$

The $\varepsilon^1$ order solutions of the second group are given by $x_{i1} = 0, i = 3, 4, \ldots, n$. We substitute $x_{i1} = 0\ (i = 3, 4, \ldots, n)$ into (29). Then (29) is divided into four groups. The first and second equations are the first group. In this group, we can get ${D_0}^2x_{12} + x_{12} = -D_1D_0x_{11} - D_1x_{21} + D_0f_{12} + f_{22}$. To eliminate possible secular terms which may appear in $x_{12}$, we find $D_1r = D_1\varphi = 0$. The third, fourth and fifth equations are the second group, and after we solve the fourth and fifth equations of (29), we can solve the third one. The sixth, seventh, eighth, and ninth equations are the third group, and after we solve the eighth and ninth equations of (29), we can solve the sixth and seventh equations. The tenth and eleventh equations are the fourth group, we can solve the group directly. In the discussion above, we solve the equations by the harmonic balance method. Finally, $x_{i2}\ (i = 1, 2, \ldots, n)$ can be obtained. This method is also applicable to higher order perturbation equations. Then, by the maple program in the computer, in the finite step, one gets $D_ir = c_1r^{i+1}, D_i\varphi = c_2r^i$, if $i$ is even, where $c_1$, $c_2$ are constants; $D_ir = D_i\theta = 0$, if is odd. The maple program mentioned above is available in [12].

**Lemma 1.** *In the expansion of* $x_i$, $x_{ij} = r^j[C_0 + \sum_{k=1}^{j}(C_{1k}\cos k\theta + C_{2k}\sin k\theta)]$, $C_0$, $C_{1k}$, $C_{2k}$ *are constants. If* $j$ *is even, then* $C_{1k} = C_{2k} = 0$, *in which* $k$ *is odd, if* $j$ *is odd, then* $C_0 = C_{1k} = C_{2k} = 0$, *in which* $k$ *is even.*

**Proof.** This lemma is proved by induction. Note that the terms like $C_{1k}\cos k\theta$, $C_{2k}\sin k\theta$ are called odd term, if k is odd; the terms like $C_{1k}\cos k\theta, C_{2k}\sin k\theta$ are called even term, if k is even. First, the method introduced above is used to solve

$$
\begin{aligned}
&x_{11} = r\cos\theta, x_{21} = -r\sin\theta, x_{31} = x_{41} = \cdots = x_{n1} = 0,\\
&x_{l2} = r^2(C_0 + C_{12}\cos 2\theta + C_{22}\sin 2\theta),\ \ l = 1, 2, \ldots, n.
\end{aligned} \tag{30}
$$

Obviously, $x_{l1}$, $x_{l2}$ are satisfying Lemma 1, and we assume that the lemma is right when $j = m-1$. We can write $\varepsilon^m$ order perturbation equations, and the first two equations of them is written, as follows,

$$
\begin{aligned}
&D_{m-1}x_{11} + D_{m-2}x_{12} + \cdots + D_0x_{1m} = x_{2m} + f_{1m},\\
&D_{m-1}x_{21} + D_{m-2}x_{22} + \cdots + D_0x_{1m} = -x_{1m} + f_{2m}.
\end{aligned} \tag{31}
$$

From (31), we find that

$$D_0^2x_{1m}+x_{1m}=-D_0D_{m-1}x_{11}-D_0D_{m-2}x_{12}-\cdots-D_0D_1x_{1(m-1)}-D_{m-1}x_{21}-D_{m-2}x_{22}\cdots-D_1x_{2(m-1)}+f_{2m}+D_0f_{1m}. \tag{32}$$

If m is odd, according to the results we have got, the terms which contain $D_i$ ($i$ is odd) in the right of (32) equal 0. Thus (32) is written as

$$D_0^2x_{1m}+x_{1m}=-D_0D_{m-1}x_{11}-D_0D_{m-3}x_{13}-\cdots-D_0D_2x_{1(m-2)}-D_{m-1}x_{21}-D_{m-3}x_{23}\cdots-D_2x_{2(m-2)}+f_{2m}+D_0f_{1m}. \tag{33}$$

We assume that

$$x_{1m}=A_0+A_1\cos\theta+B_1\sin\theta+\cdots+A_m\cos m\theta+B_m\sin m\theta. \tag{34}$$

On the right side of (33), according to the induction hypothesis, and because $D_ir=const.r^{i+1}, D_i\varphi=const.r^i$, if $i$ is even; $D_ir=D_i\theta=0$, if $i$ is odd, we find every term is like the form of $r^m[C_0+\sum_{k=1}^{m}(C_{1k}\cos k\theta+C_{2k}\sin k\theta)]$, $C_0$, $C_{1k}$, $C_{2k}$ are constants.

Because $f_{im}=(d^m/d\varepsilon^m)[f_i(x_1,x_2,\dots,x_n)]_{\varepsilon=0}(i=1,2,\dots,n)$, and m is odd, by induction hypothesis, thus the last two terms $f_{2m}$, $D_0f_{1m}$ only contain odd term, the remaining terms contain odd term obviously. Thus the right side of (33) only contains odd term. Then, substituting (34) into (33), and balancing the corresponding coefficients of both sides, we get

$$x_{1m}=r^m[A_{11}\cos\theta+B_{11}\sin\theta+A_{31}\cos 3\theta+B_{31}\sin 3\theta+\cdots+A_{m1}\cos m\theta+B_{m1}\sin m\theta]. \tag{35}$$

Substituting (35) into the first equation of (31), we get

$$x_{2m}=r^m[A_{12}\cos\theta+B_{12}\sin\theta+A_{32}\cos 3\theta+B_{32}\sin 3\theta+\cdots+A_{m2}\cos m\theta+B_{m2}\sin m\theta]. \tag{36}$$

Note that $A_{i1}, A_{i2}, B_{i1}, B_{i2}$ $(i=1,2,\dots,m)$ in (35) and (36) are constants. Similarly, according to the conditions of the induction hypothesis, from the rest equations of $\varepsilon^m$ order perturbation equations, we find that if m is odd, then

$$x_{im}=r^m[A_{1i}\cos\theta+B_{1i}\sin\theta+A_{3i}\cos 3\theta+B_{3i}\sin 3\theta+\cdots+A_{mi}\cos m\theta+B_{mi}\sin m\theta],\ \ i=1,2,\dots,n; \tag{37}$$

if m is even, then

$$x_{im}=r^m[A_{0i}+A_{2i}\cos 2\theta+B_{2i}\sin 2\theta+A_{4i}\cos 4\theta+B_{4i}\sin 4\theta+\cdots+A_{mi}\cos m\theta+B_{mi}\sin m\theta],\ \ i=1,2,\dots,n, \tag{38}$$

where $A_{ji}, B_{hi}$ $(j=0,1,2,\dots,m, h=1,2,\dots,m, i=1,2,\dots,n)$ in (37) and (38) are constants.

Thus, the Lemma 1 is obtained. In the previous section, we have obtained that

$D_ir=c_1r^{i+1}, D_i\varphi=c_2r^i$, where $c_1$, $c_2$ are constants, as $i$ is even;

$D_ir=D_i\theta=0$, as $i$ is odd, where $i$ is finite.

Next,we will show that if we keep computation, we will prove that the result is the same. □

**Lemma 2.** *In the system (27), if i is even, then $D_ir=c_1r^{i+1}, D_i\varphi=c_2r^i$, where $c_1$, $c_2$ are constants; if i is odd, then $D_ir=D_i\theta=0, i=1,2,\dots$.*

**Proof.** According to the method introduced above, to eliminate the possible secular term in $x_{1m}$, we get $D_ir$, $D_i\varphi$. Thus we can study the problem in (32), as follows,

$$D_0^2x_{1m}+x_{1m}=-D_0D_{m-1}x_{11}-D_0D_{m-2}x_{12}-\cdots-D_0D_1x_{1(m-1)}-D_{m-1}x_{21}-D_{m-2}x_{22}\cdots-D_1x_{2(m-1)}+f_{2m}+D_0f_{1m},$$

where $m=1,2,3,\dots$.

From the above equation, we can see that only terms $(-D_0D_{m-1}x_{11}-D_{m-1}x_{21})$ contains $D_{m-1}$. Then we use the induction method to solve this problem. According to the previous algorithm, we get $D_1r=D_1\varphi=0, D_2r=a_{13}r^3, D_2\varphi=a_{23}r^2$, which satisfy Lemma 2 obviously. We assume the conclusion is right when $i\le m-2$, and because

$$\begin{aligned}-D_0D_{m-1}x_{11}-D_{m-1}x_{21}&=-2D_{m-1}x_{21}\\&=2D_{m-1}r\cdot\sin\theta+2rD_{m-1}\varphi\cdot\cos\theta.\end{aligned} \tag{39}$$

From (39), we know that $D_{m-1}r$ only appears as the coefficient of $\sin\theta$, and $D_{m-1}\varphi$ only appears as the coefficient of $\cos\theta$. Thus, in the process of eliminating the secular term and solving $D_{m-1}r$, we only consider the terms about $\sin\theta$.

When m is odd, (32) can be written as

$$D_0^2x_{1m}+x_{1m}=(-D_0D_{m-1}x_{11}-D_{m-1}x_{21})-D_0D_{m-3}x_{13}-\cdots-D_0D_2x_{1(m-2)}-D_{m-3}x_{23}\cdots-D_2x_{2(m-2)}+f_{2m}+D_0f_{1m}, \tag{40}$$

then, according to Lemma 1, we know that the right of (40) can be written as

$$(2D_{m-1}r\cdot\sin\theta+2rD_{m-1}\varphi\cdot\cos\theta)+r^m[A_1\cos\theta+B_1\sin\theta+A_3\cos 3\theta+B_3\sin 3\theta+\cdots+A_m\cos m\theta+B_m\sin m\theta],$$

where $A_1, A_3, \dots, A_m, B_1, B_3, \dots, B_m$ are constants. To eliminate the secular term that may appear in $x_{1m}$, we get that $D_{m-1}r=const.r^m, D_{m-1}\varphi=const.r^{m-1}$.

When m is even, we also consider

$$D_0^2x_{1m}+x_{1m}=-D_0D_{m-1}x_{11}-D_0D_{m-2}x_{12}-\cdots-D_0D_1x_{1(m-1)}-D_{m-1}x_{21}-D_{m-2}x_{22}\cdots-D_1x_{2(m-1)}+f_{2m}+D_0f_{1m},$$

then it can be written as,

$$D_0^2x_{1m}+x_{1m}=(-D_0D_{m-1}x_{11}-D_{m-1}x_{21})-D_0D_{m-2}x_{12}-\cdots-D_0D_2x_{1(m-2)}-D_{m-2}x_{22}\cdots-D_2x_{2(m-2)}+f_{2m}+D_0f_{1m}. \tag{41}$$

According to Lemma 1, we find that the right of (41) can be written as

$$(2D_{m-1}r\cdot\sin\theta+2rD_{m-1}\varphi\cdot\cos\theta)+r^m[A_0+A_2\cos 2\theta+B_2\sin 2\theta+A_4\cos 4\theta+B_4\sin 4\theta+\cdots$$
$$+A_m\cos m\theta+B_m\sin m\theta].$$

To eliminate the secular term that may appear in $x_{1m}$, we get that $D_{m-1}r=D_{m-1}\varphi=0$.

Thus Lemma 2 is obtained.

Next, we give a back scaling $\varepsilon r\to r$ or simply setting $\varepsilon=1$, then we get

$$\begin{aligned}x_1&=r\cos\theta+h_1(r\cos\theta,-r\sin\theta),\\ x_2&=-r\sin\theta+h_2(r\cos\theta,-r\sin\theta),\\ x_i&=h_i(r\cos\theta,-r\sin\theta),\ \mathrm{i}=3,4,\ldots,n.\end{aligned} \tag{42}$$

In polar coordinates, we find that

$$\begin{aligned}\dot r&=D_2r+D_4r+D_6r+\ldots+D_{2n}r+\ldots\\ &=a_{13}r^3+a_{15}r^5+a_{17}r^7+\ldots+a_{1(2n+1)}r^{2n+1}+\ldots,\\ \dot\theta&=1+D_2\varphi+D_4\varphi+D_6\varphi+\ldots+D_{2n}\varphi+\ldots\\ &=1+a_{23}r^2+a_{25}r^4+a_{27}r^6+\ldots+a_{2(2n+1)}r^{2n}+\ldots.\end{aligned} \tag{43}$$

Similar to the study of 3-dimensional systems, one has gotten the simplest normal form of (43) in [20], and we just discuss the case that (43) can be reduced to the following form,

$$\begin{cases}\dot\rho=a_{13}\rho^3+a_{15}\rho^5,\\ \dot\varphi=1+a_{23}\rho^2,\end{cases}\qquad if\ a_{13}\neq 0; \tag{44}$$

Note that $a_{13}$ of (43) is as same as $a_{13}$ in (44). The next work is to prove that (27) can be reduced by the center manifold theory to (44). If the transformation $y_1=r\cos\theta,\ y_2=-r\sin\theta$ is introduced into (42), we can rewrite Eq. (42) as follows,

$$x_1=y_1+h_1(y_1,\ y_2),x_2=y_2+h_2(y_1,y_2),\ \ x_i=h_i(y_1,y_2),i=3,4,\ldots,n. \tag{45}$$

Let us make the following near-identity transformation:

$$x_1=y_1+h_1(y_1,y_2),\ \ x_2=y_2+h_2(y_1,y_2),\ \ x_i=y_i,i=3,4,\ldots,n. \tag{46}$$

Then,

$$\begin{pmatrix}\dot x_1\\ \dot x_2\\ \cdot\\ \cdot\\ \cdot\\ \dot x_n\end{pmatrix}=\begin{bmatrix}1+\frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & & & \\ \frac{\partial h_2}{\partial y_1} & 1+\frac{\partial h_2}{\partial y_2} & & & \\ & & 1 & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1\end{bmatrix}\begin{pmatrix}\dot y_1\\ \dot y_2\\ \cdot\\ \cdot\\ \cdot\\ \dot y_n\end{pmatrix},$$

$$\begin{pmatrix}\dot y_1\\ \dot y_2\\ \cdot\\ \cdot\\ \cdot\\ \dot y_n\end{pmatrix}=\begin{bmatrix}1+\frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & & & \\ \frac{\partial h_2}{\partial y_1} & 1+\frac{\partial h_2}{\partial y_2} & & & \\ & & 1 & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1\end{bmatrix}^{-1}\begin{pmatrix}\dot x_1\\ \dot x_2\\ \cdot\\ \cdot\\ \cdot\\ \dot x_n\end{pmatrix}$$

$$=\begin{bmatrix} 1+\frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & & & \\ \frac{\partial h_2}{\partial y_1} & 1+\frac{\partial h_2}{\partial y_2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}^{-1} \begin{pmatrix} x_2+f_1(x_1,x_2,\dots,x_n) \\ -x_1+f_2(x_1,x_2,\dots,x_n) \\ \vdots \\ -\omega_{n-1}x_{n-1}-\alpha_{n-1}x_n+f_n(x_1,x_2,\dots,x_n) \end{pmatrix}. \tag{47}$$

We can obtain the system (48) which has the same stable properties as (27) at the origin by substituting (46) into (47), as follows,

$$\begin{aligned} \dot{y}_1 &= y_2+e_1(y_1,y_2,\dots,y_n), \\ \dot{y}_2 &= -y_1+e_2(y_1,y_2,\dots,y_n), \\ \dot{y}_i &= e_i(y_1,y_2,\dots,y_n), i=3,4,\dots,n. \end{aligned} \tag{48}$$

We get $y_i = h_i(y_1, y_2) (i = 3, 4, \dots, n)$ from (45) and (46). According to center manifold theory, we can prove that $y_i = h_i(y_1, y_2), i = 3, 4, \dots, n$ is the center manifold of the n-dimensional system (48).

Substituting $y_i = h_i(y_1, y_2)$ into the first two equations of (48), we can get the reduced system ,

$$\begin{aligned} \dot{y}_1 &= y_2+a_{13}({y_1}^2+{y_2}^2)y_1+\cdots+a_{1(2n+1)}({y_1}^2+{y_2}^2)^n y_1+\cdots+a_{23}({y_1}^2+{y_2}^2)y_2+\cdots \\ &\quad +a_{2(2n+1)}({y_1}^2+{y_2}^2)^n y_2+\cdots \\ \dot{y}_2 &= -y_1+a_{13}({y_1}^2+{y_2}^2)y_2+\cdots+a_{1(2n+1)}({y_1}^2+{y_2}^2)^n y_2+\cdots-a_{23}({y_1}^2+{y_2}^2)y_1-\cdots \\ &\quad -a_{2(2n+1)}({y_1}^2+{y_2}^2)^n y_1-\cdots. \end{aligned} \tag{49}$$

And the reduced system is obtained in [11–13] too. The Eqs. (49) given in polar co-ordinates can be written as (43). Thus the system (48) and (49)(43)have the same stability properties at the origin. And the system (48) and (27) have the same stability properties at the origin, so the system (27) and (43) have the same stability properties at the origin.

To give the following theorem for determining the stability of n-dimensional system conveniently, $f_k$ in (27) is written as

$$\begin{aligned} f_1 &= s_{11}{x_1}^2+s_{12}{x_2}^2+s_{13}x_1x_2+s_{14}x_1x_3+s_{15}x_2x_3+s_{16}{x_1}^3+f_{11}, \\ f_2 &= s_{21}{x_1}^2+s_{22}{x_2}^2+s_{23}x_1x_2+s_{24}x_1x_3+s_{25}x_2x_3+s_{26}{x_2}^3+f_{21}, \\ f_k &= s_{k1}{x_1}^2+s_{k2}{x_2}^2+s_{k3}x_1x_2+f_{k1}, k=3,4,\dots,n, \end{aligned} \tag{50}$$

where $f_{11}$ does not contain terms of ${x_1}^2$, ${x_2}^2$, $x_1x_2$, $x_1x_3$, $x_2x_3$, ${x_1}^3$, $f_{21}$ does not contain terms of ${x_1}^2$, ${x_2}^2$, $x_1x_2$, $x_1x_3$, $x_2x_3$, ${x_2}^3$, $f_{k1}$ does not contain terms of ${x_1}^2$, ${x_2}^2$, $x_1x_2$. And we assume $x_{i2} = B_{i0}r^2 + B_{i1}r^2\cos 2\theta + B_{i2}r^2\sin 2\theta$. □

**Theorem 2.** *For the system (27), if $\Delta < 0$, then the system will be unstable at the origin; if $\Delta > 0$, then the system will be asymptotically stable at the origin, where*

$$\begin{aligned} \Delta &= \sum_{i=1}^{n}\left(\frac{1}{2}\xi_{i2}B_{i2}-\xi_{i1}\left(B_{i0}+\frac{1}{2}B_{i1}\right)\right)+\sum_{j=1}^{n}\left(\frac{1}{2}\tau_{j1}B_{j2}-\tau_{j2}\left(B_{j0}-\frac{1}{2}B_{j1}\right)\right)+\frac{1}{2}(B_{22}\xi_{22}+B_{12}\tau_{11})-\left(B_{10}+\frac{1}{2}B_{11}\right)\xi_{11} \\ &\quad -\left(B_{20}-\frac{1}{2}B_{21}\right)\tau_{22}-\frac{3}{4}(s_{16}+s_{26}), \end{aligned}$$

*$\xi_{i1}$ and $\xi_{i2}$ are the coefficients of $x_1x_i$ and $x_2x_i$ in $f_1$ respectively; $\tau_{j1}$ and $\tau_{j2}$ are the coefficients of $x_1x_j$ and $x_2x_j$ in $f_2$ respectively; $B_{i0}$, $B_{i1}$, $B_{i2}$ are defined by $x_{i2} = B_{i0}r^2 + B_{i1}r^2\cos 2\theta + B_{i2}r^2\sin 2\theta$, in which*

$$B_{10}=\frac{s_{21}+s_{22}}{2}, B_{11}=\frac{s_{22}+2s_{13}-s_{21}}{6}, B_{12}=\frac{s_{23}+2s_{11}-2s_{12}}{6},$$
$$B_{20}=-\frac{s_{11}+s_{12}}{2}, B_{21}=\frac{2s_{23}+s_{11}-s_{12}}{6}, B_{22}=\frac{2s_{21}-2s_{22}-s_{13}}{6},$$
$$B_{a0}=\frac{1}{\alpha_1}B_{a+1,0}+\frac{s_{a1}+s_{a2}}{2\alpha_1}, B_{a1}=\frac{\alpha_1 B_{a+1,1}-2B_{a+1,2}}{\alpha_1{}^2+4}+\frac{\alpha_1 s_{a1}-\alpha_1 s_{a2}+2s_{a3}}{2(\alpha_1{}^2+4)},$$
$$B_{a2}=\frac{2B_{a+1,1}+\alpha_1 B_{a+1,2}}{\alpha_1{}^2+4}+\frac{2s_{a1}-2s_{a2}-\alpha_1 s_{a3}}{2(\alpha_1{}^2+4)}, a=3,4,\dots,k_1+2,$$
$$B_{b0}=\frac{s_{b1}+s_{b2}}{2\alpha_1}, B_{b1}=\frac{\alpha_1 s_{b1}-\alpha_1 s_{b2}+2s_{b3}}{2(\alpha_1{}^2+4)}, B_{b2}=\frac{2s_{b1}-2s_{b2}-\alpha_3 s_{b3}}{2(\alpha_1{}^2+4)}, b=k_1+3,\dots,k_2+2,$$
$$B_{p0}=\frac{s_{p1}+s_{p2}}{2\alpha_p}, B_{p1}=\frac{\alpha_p s_{p1}-\alpha_p s_{p2}+2s_{p3}}{2(\alpha_p{}^2+4)}, B_{p2}=\frac{2s_{p1}-2s_{p2}-\alpha_3 s_{p3}}{2(\alpha_p{}^2+4)}, p=k_2+3,\dots,k_3,$$
$$B_{c0}=\frac{\alpha B_{c+2,0}+\omega B_{c+3,0}}{\omega^2+\alpha^2}+\frac{\alpha(s_{c1}+s_{c2})+\omega(s_{c+1,1}+s_{c+1,2})}{2(\omega^2+\alpha^2)},$$
$$B_{c1}=\frac{-4\alpha v_{c2}+(\alpha^2+\omega^2-4)v_{c1}}{16\alpha^2+(\alpha^2+\omega^2-4)^2}, B_{c2}=\frac{4\alpha v_{c1}+(\alpha^2+\omega^2-4)v_{c2}}{16\alpha^2+(\alpha^2+\omega^2-4)^2},$$
$$v_{c1}=2B_{c+2,2}+\omega B_{c+3,1}+\alpha B_{c+2,1}-s_{c3}+\frac{\alpha(s_{c1}-s_{c2})+\omega(s_{c+1,1}-s_{c+1,2})}{2},$$
$$v_{c2}=-2B_{c+2,1}+\omega B_{c+3,2}+\alpha B_{c+2,2}+s_{c2}-s_{c1}-\frac{1}{2}(\alpha s_{c3}+\omega s_{c+1,3}),$$
$$c=k_3+1,k_3+3,\dots,k_4-1,$$
$$B_{d0}=\frac{\omega(s_{d+1,1}+s_{d+1,2})+\alpha(s_{d1}+s_{d2})}{2(\omega^2+\alpha^2)}, B_{d1}=\frac{-4\alpha u_{d2}+(\alpha^2+\omega^2-4)u_{d1}}{16\alpha^2+(\alpha^2+\omega^2-4)^2}, B_{d2}=\frac{4\alpha u_{d1}+(\alpha^2+\omega^2-4)u_{d2}}{16\alpha^2+(\alpha^2+\omega^2-4)^2},$$
$$u_{d1}=-s_{d3}+\frac{\alpha(s_{d1}-s_{d2})+\omega(s_{d+1,1}-s_{d+1,2})}{2}, u_{d2}=s_{d2}-s_{d1}-\frac{1}{2}(\alpha s_{d3}+\omega s_{d+1,3}),$$
$$B_{d+1,0}=\frac{-\omega(s_{d1}+s_{d2})+\alpha(s_{d+1,1}+s_{d+1,2})}{2(\omega^2+\alpha^2)}, B_{d+1,1}=\frac{1}{\omega}\left(-2B_{d1}+\alpha B_{d2}+\frac{1}{2}s_{d3}\right),$$
$$B_{d+1,2}=\frac{1}{\omega}\left(2B_{d2}+\alpha B_{d1}-\frac{s_{d1}-s_{d2}}{2}\right),$$
$$d=k_4+1,k_4+3,\dots,k_5-1,$$
$$B_{q0}=\frac{\omega_q(s_{q+1,1}+s_{q+1,2})+\alpha_q(s_{q1}+s_{q2})}{2(\omega_q{}^2+\alpha_q{}^2)}, B_{q1}=\frac{-4\alpha_q u_{q2}+(\alpha_q{}^2+\omega_q{}^2-4)u_{q1}}{16\alpha_q{}^2+(\alpha_q{}^2+\omega_q{}^2-4)^2}, B_{q2}=\frac{4\alpha_q u_{q1}+(\alpha_q{}^2+\omega_q{}^2-4)u_{q2}}{16\alpha_q{}^2+(\alpha_q{}^2+\omega_q{}^2-4)^2},$$
$$u_{q1}=-s_{q3}+\frac{\alpha_q(s_{q1}-s_{q2})+\omega_q(s_{q+1,1}-s_{q+1,2})}{2}, u_{q2}=s_{q2}-s_{q1}-\frac{1}{2}(\alpha_q s_{q3}+\omega_q s_{q+1,3}),$$
$$B_{q+1,0}=\frac{-\omega_q(s_{q1}+s_{q2})+\alpha_q(s_{q+1,1}+s_{q+1,2})}{2(\omega_q{}^2+\alpha_q{}^2)}, B_{q+1,1}=\frac{1}{\omega_q}\left(-2B_{q1}+\alpha_q B_{q2}+\frac{1}{2}s_{q3}\right),$$
$$B_{q+1,2}=\frac{1}{\omega_q}\left(2B_{q2}+\alpha_q B_{q1}-\frac{s_{q1}-s_{q2}}{2}\right),$$
$$q=k_5+1,k_5+3,\dots,n-1.$$

**Proof.** We need to judge the stability of (43), firstly, we need to solve $a_{13}$. Because $a_{13}r^3=D_2r$, then the next work is to solve $D_2r$. According to the method introduced above, in the $\varepsilon^3$ order perturbation equations, to eliminate the possible secular term in $x_{13}$, we can solve $D_2r$. From the first two equations in the $\varepsilon^3$ order perturbation equations, $D_0x_{13}+D_1x_{12}+D_2x_{11}=x_{23}+f_{13}, D_0x_{23}+D_1x_{22}+D_2x_{21}=-x_{13}+f_{23}$, we obtain that

$$D_0{}^2x_{13}+x_{13}=-D_0D_1x_{12}-D_0D_2x_{11}-D_1x_{22}-D_2x_{21}+f_{23}+D_0f_{13}.$$

Substituting $x_{11}$, $x_{21}$ into the above equation, we obtain

$$D_0{}^2x_{13}+x_{13}=2(D_2r\cdot\sin\theta+rD_2\varphi\cdot\cos\theta)+f_{23}+D_0f_{13}. \tag{51}$$

According to the definitions above, $f_{i3}(i=1,2,\dots,n)$ is in terms of $x_{11}{}^3, x_{21}{}^3$, $x_{11}x_{i2}, x_{21}x_{i2}$, $i=1,2,\dots,n$. Because $D_2r$ only appears as the coefficient of $\sin\theta$, we only need to consider the coefficients of $\sin\theta$ in $f_{23}+D_0f_{13}$. Thus we only consider the terms like $x_{11}x_{i2}, x_{21}x_{i2}, x_{11}{}^3(i=1,2,\dots,n)$ in $f_{13}$, and consider the terms like $x_{11}x_{i2}, x_{21}x_{i2}, x_{21}{}^3(i=1,2,\dots,n)$ in $f_{23}$. According to the lemmas we have given, we assume $x_{i2}=B_{i0}+B_{i1}\cos 2\theta+B_{i2}\sin 2\theta$, and substituting the above equation into (51), we find

$$D_0^2x_{13}+x_{13} = 2(D_2r\cdot\sin\theta+rD_2\varphi\cdot\cos\theta)+\Bigg[\sum_{i=1}^{n}\left(\frac{1}{2}\xi_{i2}B_{i2}-\xi_{i1}(B_{i0}+\frac{1}{2}B_{i1})\right)+\sum_{j=1}^{n}\left(\frac{1}{2}\tau_{j1}B_{j2}-\tau_{j2}\left(B_{j0}-\frac{1}{2}B_{j1}\right)\right)$$
$$+\frac{1}{2}(B_{22}\xi_{22}+B_{12}\tau_{11})-\left(B_{10}+\frac{1}{2}B_{11}\right)\xi_{11}-\left(B_{20}-\frac{1}{2}B_{21}\right)\tau_{22}-\frac{3}{4}(s_{16}+s_{26})\Bigg]r^3\sin\theta+g, \quad (52)$$

where $g$ does not contain the terms like $\sin\theta$.

To eliminate the possible secular term in $x_{13}$, in the right of (52), it is required that the coefficients of $\sin\theta$, $\cos\theta$ equal 0, which in turn yields

$$D_2r = -\frac{1}{2}r^3\Bigg[\sum_{i=1}^{n}\left(\frac{1}{2}\xi_{i2}B_{i2}-\xi_{i1}\left(B_{i0}+\frac{1}{2}B_{i1}\right)\right)+\sum_{j=1}^{n}\left(\frac{1}{2}\tau_{j1}B_{j2}-\tau_{j2}\left(B_{j0}-\frac{1}{2}B_{j1}\right)\right)+\frac{1}{2}(B_{22}\xi_{22}+B_{12}\tau_{11})$$
$$-\left(B_{10}+\frac{1}{2}B_{11}\right)\xi_{11}-\left(B_{20}-\frac{1}{2}B_{21}\right)\tau_{22}-\frac{3}{4}(s_{16}+s_{26})\Bigg]. \quad (53)$$

Thus the next task is to get $B_{i0}$, $B_{i1}$, $B_{i2}$, and solve $x_{i2}$ in the $\varepsilon^2$ order perturbation Eq. (29). In the first and second equations, according to the method for studying 3-dimensional system, by using the method of harmonic balance, we get $B_{10}$, $B_{11}$, $B_{12}$, $B_{20}$, $B_{21}$, $B_{22}$, $x_{12}$, $x_{22}$. In the rest of the Eq. (29), $B_{j0}, B_{j1}, B_{j2}(j=3,4,\ldots,n)$ are obtained. Finally, we find that $a_{13}=-\frac{1}{2}\Delta$.

In the system (43), if $a_{13}>0$, $(\Delta<0)$, then the system is unstable; if $a_{13}<0, (\Delta>0)$, then the system is asymptotically stable. And because the system (43) and (27) have the same stability properties at the origin, thus the Theorem 2 is obtained. □

## 4. Examples

In this section we shall present several examples and apply the theorems above in these examples. The first example, which is 3-dimensional system, is described by the following differential equations:

$$\begin{aligned}\dot{x}_1 &= x_2+{x_1}^2-x_1x_3+{x_1}^3+2{x_1}^2x_2+3{x_2}^4,\\ \dot{x}_2 &= -x_1+{x_2}^2+x_1x_3+{x_2}^3+x_1{x_2}^2+2{x_3}^4,\\ \dot{x}_3 &= -2x_3+{x_1}^2+{x_2}^2+{x_1}^4.\end{aligned} \quad (54)$$

(It should be noted that the coefficients of vector field given in (54) are not necessary integers.) It is seen that the origin $x=0$ is an equilibrium; and the linearized system of (54) has eigenvalues $\pm i$ and $-2$ at the origin. According to the Theorem 1, we have

$$p_1=1, p_2=0, p_3=0, p_4=-1, p_5=0, p_6=1, q_1=0, q_2=1, q_3=0,$$

$$q_4=1, q_5=0, q_6=1, \alpha_3=2, r_1=1, r_2=1, r_3=0,$$

then we can obtain the result :

$$\Delta = (p_3-2q_1)p_1+(p_3+2q_2)p_2-(q_1+q_2)q_3-3(p_6+q_6)+\frac{(p_5+q_4)(2r_1-2r_2-\alpha_3r_3)}{{\alpha_3}^2+4}$$
$$+\frac{(-p_4+q_5)(\alpha_3r_1-\alpha_3r_2+2r_3)}{{\alpha_3}^2+4}-\frac{2(p_4+q_5)(r_1+r_2)}{\alpha_3}=-4<0,$$

and the system (54) is not stable.

The second example, which is 5-dimensional system and has been considered in [11–13], is described by the following differential equations:

$$\begin{aligned}\dot{x}_1 &= x_2+{x_1}^2-x_1x_3+{x_2}^3,\\ \dot{x}_2 &= -x_1+{x_2}^2+x_1x_4+{x_1}^3+{x_2}^3+{x_2}^4,\\ \dot{x}_3 &= -x_3+{x_1}^2+{x_3}^3,\\ \dot{x}_4 &= -x_4+x_5+{x_1}^2+{x_3}^2+{x_4}^5,\\ \dot{x}_5 &= -x_4-x_5+{x_2}^2+{x_4}^2+{x_5}^3.\end{aligned} \quad (55)$$

It is seen that the origin $x = 0$ is an equilibrium; and the linearized system of (54) has eigenvalues $\pm i$, $-1$ and $-1 \pm i$ at the origin. According to the Theorem 2, we have

$$\begin{aligned}
&s_{11} = 1, s_{12} = 0, s_{13} = 0, s_{14} = -1, s_{15} = 0, s_{16} = 0, s_{21} = 0, s_{22} = 1, s_{23} = 0, s_{24} = 0,\\
&s_{25} = 0, s_{26} = 1, s_{31} = 1, s_{32} = 0, s_{33} = 0, s_{41} = 1, s_{42} = 0, s_{43} = 0, s_{51} = 0, s_{52} = 1, s_{53} = 0,\\
&\xi_{12} = \xi_{22} = \xi_{32} = \xi_{42} = \xi_{52} = 0, \xi_{11} = 1, \xi_{21} = 0, \xi_{31} = -1, \xi_{41} = 0, \xi_{51} = 0,\\
&\tau_{11} = \tau_{21} = \tau_{31} = \tau_{51} = 0, \tau_{41} = 1, \tau_{12} = \tau_{32} = \tau_{42} = \tau_{52} = 0, \tau_{22} = 1,\\
&B_{12} = \frac{1}{3}, B_{22} = -\frac{1}{3}, B_{32} = \frac{1}{5}, B_{42} = \frac{1}{10}, B_{52} = -\frac{1}{10}, B_{11} = \frac{1}{6}, B_{21} = \frac{1}{6},\\
&B_{31} = \frac{1}{10}, B_{41} = \frac{1}{5}, B_{51} = -\frac{3}{10}, B_{10} = \frac{1}{2}, B_{20} = -\frac{1}{2}, B_{30} = \frac{1}{2}, B_{40} = \frac{1}{2}, B_{50} = 0,
\end{aligned}$$

then we can obtain the result:

$$\begin{aligned}
\Delta = &\sum_{i=1}^{5}\left(\frac{1}{2}\xi_{i2}B_{i2} - \xi_{i1}\left(B_{i0} + \frac{1}{2}B_{i1}\right)\right) + \sum_{j=1}^{5}\left(\frac{1}{2}\tau_{j1}B_{j2} - \tau_{j2}\left(B_{j0} - \frac{1}{2}B_{j1}\right)\right) + \frac{1}{2}(B_{22}\xi_{22} + B_{12}\tau_{11}) - \left(B_{10} + \frac{1}{2}B_{11}\right)\xi_{11}\\
&-\left(B_{20} - \frac{1}{2}B_{21}\right)\tau_{22} - \frac{3}{4}(s_{16} + s_{26}) = -\frac{3}{20} < 0,
\end{aligned}$$

and the system (55) is not stable.

Obviously, with the help of Maple, the method in this paper is more simple and practical than the traditional Lyapunov method which needs a suitable Lyapunov function.

## 5. Conclusions

We first give a theorem to determine the stability of the 3-dimensional system, then we solve the center manifold and normal form of n-dimensional system. At last, the discrimination method for the stability of n-dimensional system is given. The Jacobian matrix of the system studied here has only a pair of pure imaginary eigenvalues, and the other eigenvalues are hyperbolic, so this is Hopf bifurcation case. In the above theorem, we can determine the stability by simple calculation of a discriminant, and this discriminant can be calculated by coefficients of the original system. In the study of 3-dimensional or n-dimensional system above, we only consider the situation that $a_{13}$ in the simplest normal form does not equal 0, and if $a_{13} = 0$, the calculation will be more complicated, so it needs further study. Further we can also consider the stability of the system in which the eigenvalues of Jacobian matrix are in other cases, and give a corresponding discriminant for judging the stability.

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